Theta series

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$Q(x_1, \dots, x_n)$ positive-definite $\mathbb{Z}$-valued quadratic form.

$$\Theta_Q(z) = \sum_{\alpha \in \mathbb{Z}^m} q^{Q(\alpha)} = \sum_{n=0}^{\infty} R_Q(n) q^n \qquad q = e^{2\pi i z}$$

$$R_Q(n) = \# \{ \alpha \in \mathbb{Z}^m \mid Q(\alpha) = n \}$$

e.g $Q(x) = x^2$ Jacobi theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

Prpn

$$\theta(z+1) = \theta(z) \quad (1)$$

$$\theta\left(\frac{-1}{4z}\right) = \sqrt{\frac{2z}{i}}\, \theta(z) \quad (2)$$

proof (1) clear

(2) Poisson transformation formula

f Schwartz function $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \tilde{f}(n)$

$$\tilde{f}(y) = \int_{-\infty}^{\infty} e^{2\pi i x y} f(x)\, dx.$$

Apply to $f(x) = e^{-\pi t x^2}$, $t > 0$

$$\tilde{f}(y) = \frac{1}{\sqrt{t}} e^{-\pi y^2 / t}$$

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / t}$$

proves (2) for $z = \frac{it}{2}$, positive imaginary axis.

Use analytic continuation $\square$

$z \mapsto z+1$, $z \mapsto \frac{-1}{4z}$ generate $\Gamma_0^+(4) = \Gamma_0(4) \cup \Gamma_0(4) W_4$

$$W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \qquad \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \,\middle|\, N \mid c \right\}$$

Say "$\theta(z)$ is a modular form of weight $\frac{1}{2}$ on $\Gamma_0(4)$

$\theta(z)^2 \in M_1(\Gamma_1(4))$, $\Gamma_1(4) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \mid a \equiv 1 \pmod 4 \}$

Prpn $\Gamma \subset SL(2,\mathbb{R})$, $V = vol(\Gamma \backslash \mathcal{H}) < \infty$

$\Rightarrow \dim M_k(\Gamma) \leq \frac{kV}{4\pi} + 1$

$vol(\Gamma_1(4) \backslash \mathcal{H}) = 2\pi$

$\Rightarrow \dim M_1(\Gamma_1(4)) = 1$

$$\theta(z)^2 = \left(\sum_{a\in\mathbb{Z}} q^{a^2}\right)\left(\sum_{b\in\mathbb{Z}} q^{b^2}\right) = \sum_{n=0}^{\infty} r_2(n) q^n$$

$$r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 \mid a^2+b^2 = n\}$$

$$G_{1,\chi_{-4}}(z) = \frac{1}{4} + \sum_{n=1}^{\infty}\left(\sum_{d|n} \chi_{-4}(d)\right) q^n \in M_1(\Gamma_1(4))$$

$$\chi_{-4}(d) = \begin{cases} +1, & d \equiv 1 \pmod 4 \\ -1, & d \equiv 3 \pmod 4 \\ 0, & d \text{ even} \end{cases}$$

$$\theta(z)^2 = 4 G_{1,\chi_{-4}}(z)$$

$$r_2(n) = 4 \sum_{d|n} \chi_{-4}(d)$$

Prpn $n \in \mathbb{Z}_{>0}$, $r_2(n) = 4 \sum_{\substack{d|n \\ d>0 \text{ odd}}} (-1)^{(d-1)/2}$

Cor (Fermat) Every prime $p \equiv 1 \pmod 4$ is a sum of two squares.

proof $r_2(p) = 4(1 + (-1)^{(p-1)/4}) = 8 > 0$ $\square$

$$\theta(z)^4 = \sum_{n=0}^{\infty} r_4(n) q^n \in M_2(\Gamma_1(4))$$

$$r_4(n) = \#\{(a,b,c,d) \in \mathbb{Z}^4 \mid a^2+b^2+c^2+d^2 = n\}$$

$\dim M_2(\Gamma_1(4)) \leq 2$.

$G_2(z) - 2G_2(2z)$, $G_2(2z) - 2G_2(4z)$

$$G_2(z) = \frac{-1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

$$\sigma_k(n) = \sum_{d \mid n} d^k$$

$$\theta(z)^4 = 1 + 8q + \cdots$$

$$\theta(z)^4 = 8(G_2(z) - 2G_2(2z)) + 16(G_2(2z) - 2G_2(4z))$$

<u>prpn</u> $n \in \mathbb{Z}_{>0}$, $r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d \\ d > 0}} d$

<u>Cor</u> (Lagrange). Every positive integer is a sum of 4 squares.

$Q(x) = \frac{1}{2} x^t A x$, $x = (x_1, \dots, x_m)$

A symmetric $m \times m$ matrix.

Q integral (has integer coefficient)

$\iff$ A even integral matrix.

(integer entries, diagonal entries even).

Defn • Level of Q := smallest positive integer N such that NA^{-1} is even integral

• $\Delta := (-1)^m \det A$

$\exists!$ dirichlet character modulo N

$\chi_\Delta(p) = \left(\frac{\Delta}{p}\right)$ for $p \nmid N$ odd prime.

Thm (Hecke, Schoenburg). Let $Q : \mathbb{Z}^{2k} \Rightarrow \mathbb{Z}$ be a positive definite quadratic form level N, discriminant Δ.

θ_Q is a modular form on $\Gamma_0(N)$ of weight k character χ_Δ i.e.

$$\theta_Q\left(\frac{az+b}{cz+d}\right) = \chi_\Delta(a)(cz+d)^k \theta_Q(z)$$

$$z \in \mathcal{H}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

proof uses Poisson transformation formula.

$Q : \Lambda \Rightarrow \mathbb{Z}$, Λ $\mathbb{Z}$-module

$(x, y) = Q(x+y) - Q(x) - Q(y)$ is bilinear.

choose a lattice $\Lambda \subset \mathbb{R}^m$ $\cancel{(x,x) = \|x\|^2 = x_1^2 + \dots + x_m^2}$

we say Λ is unimodular when $Q(x) = \frac{1}{2}\|x\|^2$ is unimodular.

$Q(x) = \frac{1}{2} x^t A x$

If A even integral unimodular then A^{-1} even integral

so theorem $\Rightarrow$ $\theta_Q \in M_k(\Gamma_1)$ $\quad \Gamma_0(1) = \Gamma_1 = SL(2,\mathbb{Z})$.

<u>Prpn</u> Let $Q: \mathbb{Z}^m \to \mathbb{Z}$ positive definite even integral unimodular form. Then

i) $8 \mid m$

ii) As $n \to \infty$,

$$R_Q(n) = \frac{-2k}{B_k} \sigma_{k-1}(n) + O(n^{k/2})$$

$\#\{\alpha \in \mathbb{Z}^m \mid Q(\alpha) = n\}$

$k = \frac{m}{2}$, B_k kth Bernoulli number.